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Determining finite graphs by their large Whitney levels ☆

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Abstract

Let G be a finite connected graph and $C(G)$ the hyperspace of all subcontinua of G. A Whitney map is a continuous function $\mu : C(G) \to [0,1]$ such that $\mu(\{p\}) = 0$ for each $p \in G$, $\mu(G) = 1$ and $A \subset B \neq A$ implies that $\mu(A) < \mu(B)$. A large Whitney level is a set of the form $\mu^{-1}(t)$ where $1 > t > \max\{\mu(S)$: S is a proper nonempty connected subgraph of $G\}$. In this paper, we prove the following:

Theorem. *Let G and H be finite connected graphs with no cut points, then G and H are isomorphic if and only if large Whitney levels for $C(G)$ are homeomorphic to large Whitney levels for $C(H)$.*

Keywords: Hyperspaces; Finite graphs; Whitney levels

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0. Introduction

Throughout this paper, G and H will denote finite connected graphs. Let $C(G)$ be the hyperspace of all subcontinua of G, endowed with the Hausdorff metric $\mathscr{H}$. We denote by SG(G) the set of nonempty proper connected subgraphs of G which are not one-point sets. A *map* is a continuous function. The closed unit interval in

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the real line is denoted by I. A *Whitney map* is a continuous function $\mu: C(G) \to I$ for which $\mu(\{p\}) = 0$ for each $p \in G$, $\mu(G) = 1$ and $A \subset B \neq A$ implies that $\mu(A) < \mu(B)$. A *Whitney level* is a set of the form $\mu^{-1}(t)$, where $0 \leqslant t \leqslant 1$. A *large* (respectively a *small*) *Whitney level* is a set of the form $\mu^{-1}(t)$ where $1 > t > \max\ \mu(\mathrm{SG}(G))$ (respectively, $0 < t < \min\ \mu(\mathrm{SG}(G))$).

In [8], Kato proved that Whitney levels of finite graphs are polyhedra. He studied large Whitney levels for some particular graphs in [8, 2.6] and [10, Example 3.8]. By [6, p. 15], it follows that all the large (respectively, small) Whitney levels for $C(G)$ are homeomorphic. It follows from [8, 2.3] that small Whitney levels for $C(G)$ have the same homotopy type of G. In fact, in order to construct a small Whitney level for $C(G)$, take, for each vertex v of G, an m-simplex, where $m+1$ is the order of v in G. Then join these simplexes by segments exactly in the same way that vertices are joined in G. From this, G is completely determined by its small Whitney levels. Whitney levels for $C(G)$ different from the small ones are not so easy to construct. However, large Whitney levels contain many information about the graph as we show in the main theorem of this paper:

Theorem 0.1. *Suppose G and H have no cut points, then G and H are isomorphic graphs if and only if large Whitney levels for $C(G)$ are homeomorphic to large Whitney levels for $C(H)$.*

1. Preliminary results

We will need the following conventions: The *vertices* of G are the extremes of the segments of G. Notice that the set $\mathrm{SG}(G)$ depends on the choice of the segments. We are interested in having as few subgraphs as possible, so we will suppose that G is not a simple closed curve and each vertex of G is either an endpoint of G or a ramification point of G. With this restriction two extremes of a segment of G may coincide and then such a "segment" would be a simple closed curve. We also assume that the metric d in G is the metric of arc length and each segment has length equal to one. For each segment J in G we identify J with a closed interval $[(0)_J, (1)_J]$. Notice that it is possible that $(0)_J = (1)_J$. We write 0 (respectively 1) instead of $(0)_J$ (respectively $(1)_J$) if it causes no confusion. An acyclic connected subgraph of G which contains all the vertices of G, will be named *maximal fine subgraph* of G.

If $A \in C(G)$ and $\varepsilon > 0$, define $Q(\varepsilon, A) = \{x \in G$: there exists $a \in A$ such that $d(a, x) \leqslant \varepsilon\}$ and $N(\varepsilon, A) = \{x \in G$: there exists $a \in A$ such that $d(a, x) < \varepsilon\}$.

Throughout this paper, μ will denote a fixed Whitney map for $C(G)$. For a nonempty proper connected subgraph S of G, define $\mathfrak{M}_S = \{B \in C(G)$: $S \subset B \subset Q(1, S)$ and $B \cap N(1, S)$ is connected$\}$ and $\mathfrak{M}_S(t) = \mathfrak{M}_S \cap \mu^{-1}(t)$, if $\mu(S) < t < \mu(Q(1, S))$. Let $I_1, \dots, I_r$ be the segments in G such that, for each i, I_i intersects S in exactly one of its extremes. Let $J_1, \dots, J_s$ be the segments in G such that, for each j, J_j intersects S in exactly the two extremes of J_j (here, the two extremes of J_j can agree). Then an element A in $\mathfrak{M}_S(t)$ can be written in the form: $A = S \cup (\bigcup\{[0, a_i]: 1 \leqslant i \leqslant r\}) \cup (\bigcup\{[0, c_j] \cup [d_j, 1]: 1 \leqslant j \leqslant s\})$, where $0 \leqslant a_i \leqslant 1$ for each i, and $0 \leqslant c_j \leqslant d_j \leqslant 1$ for each j. We say that A is in the *relative interior* of $\mathfrak{M}_S(t)$ ($A \in \mathrm{RI}(\mathfrak{M}_S(t))$) if $0 < a_i < 1$ for each i, and $0 < c_j < d_j < 1$ for each j. If A does not belong to the relative interior, we say that A is in the *relative boundary* ($\mathrm{RB}(\mathfrak{M}_S(t))$) of $\mathfrak{M}_S(t)$ (that is, $\mathrm{RB}(\mathfrak{M}_S(t)) = \mathfrak{M}_S(t) - \mathrm{RI}(\mathfrak{M}_S(t))$).

Let $n = 2s + r$. Duda proved in [1] that $\mathfrak{M}_S$ is homeomorphic to I^n. As was pointed by Kato in [8, 2.4], a similar result for $\mathfrak{M}_S(t)$ holds. We extend this result in the following easy to prove theorem:

Theorem 1.1. *Let $n = 2s + r$, then:*

(a) *$\mathfrak{M}_S(t)$ is homeomorphic to I^{n-1}, and homeomorphisms from $\mathfrak{M}_S(t)$ to I^{n-1} send $\mathrm{RB}(\mathfrak{M}_S(t))$ onto the boundary of I^{n-1}.*

(b) *If $1 \leqslant r_1 \leqslant r_2 \leqslant r$ and $1 \leqslant s_1 \leqslant s_2 \leqslant s$, then the dimension of the set $\{A \in \mathfrak{M}_S(t)$: $I_1 \cup \cdots \cup I_{r_1} \subset A$, $a_{r_1+1} = 0, \ldots, a_{r_2} = 0$, $J_1 \cup \cdots \cup J_{s_1} \subset A$ and $d_{s_1+1} = 1, \ldots, d_{s_2} = 1\}$ is not greater than $n - 1 - r_2 - 2s_1 - (s_2 - s_1) = n - 1 - r_2 - s_1 - s_2$.*

Theorem 1.2. *If S is not a one-point set and $\mathrm{RI}(\mathfrak{M}_S(t))$ is nonempty, then the following assertions are equivalent:*

(a) *$\mathrm{RI}(\mathfrak{M}_S(t))$ has nonempty interior in $\mu^{-1}(t)$,*

(b) *$\mathrm{RI}(\mathfrak{M}_S(t))$ is open in $\mu^{-1}(t)$, and*

(c) *S is acyclic and S has no endpoints of G.*

Proof. (a) $\Rightarrow$ (c) Suppose that S contains a simple closed curve C. Let q be a point in C which is not a vertex of G and let J be the unique segment of G which contains q. Put $J^* = J - \{(0)_J, (1)_J\}$, then J^* is an open subset of G and $J \subset C$. It is easy to check that if $B \in C(G)$ and $S \subset B$, then $B - J^*$ is connected.

Take an arbitrary element $A \in \mathrm{RI}(\mathfrak{M}_S(t))$. Since $\mu(A) < 1$, we can enlarge A a little to obtain an element $B \in C(G)$ such that $A \subset B$ (see [12, Theorem 1.8]). A small open subinterval J_0 of J^* can be chosen such that $q \in J_0$ and $\mu(B - J_0) = t$. Since $B - J^*$ is connected, then $B - J_0$ is also connected. Therefore, $B - J_0$ is an element of $\mu^{-1}(t)$ close to A and S is not contained in $B - J_0$. This proves that A is not an interior point of $\mathrm{RI}(\mathfrak{M}_S(t))$. Hence, the interior of $\mathrm{RI}(\mathfrak{M}_S(t))$ in $\mu^{-1}(t)$ is empty.

If S has endpoints of G, a similar proof shows that the interior of $\mathrm{RI}(\mathfrak{M}_S(t))$ in $\mu^{-1}(t)$ is empty.

(c) $\Rightarrow$ (b) Take $A \in \mathrm{RI}(\mathfrak{M}_S(t))$, then A is of the form: $A = S \cup (\bigcup\{[0, a_i]: 1 \leqslant i \leqslant r\}) \cup (\bigcup\{[0, c_j] \cup [d_j, 1]: 1 \leqslant j \leqslant s\})$, where $0 < a_i < 1$ for each i, and $0 < c_j < d_j < 1$ for each j. Then $A \in N(1, S)$. Let $\varepsilon = \min(\{a_i, \ldots, a_r\} \cup \{1 - a_1, \ldots, 1 - a_r\} \cup \{c_1, \ldots, c_s\} \cup \{1 - d_1, \ldots, 1 - d_s\} \cup \{(d_1 - c_1)/2, \ldots, (d_s - c_s)/2\})$, then $N(\varepsilon, A) \subset N(1, S)$.

Let $B \in \mu^{-1}(t)$ be such that $\mathscr{H}(A, B) < \varepsilon$. We will show that $B \in \mathrm{RI}(\mathfrak{M}_S(t))$. If $1 \leqslant i \leqslant r$, from the choice of ε, $B \cap I_i$ is nonempty and $1_{I_i} \notin B$. Then $B \cap I_i = [e_i, f_i]$ for some $f_i < 1$. If $0 < e_i$, $B = [e_i, f_i]$. Since $\varepsilon < 1$ and S is a nonempty connected subgraph of G contained in $N(\varepsilon, B)$, then $S = \{(0)_{I_i}\}$, which is a contradiction. Hence $B \cap I_i = [0, f_i]$. For $1 \leqslant j \leqslant s$, $B \cap J_j$ is of the form $B \cap J_j = [0, g_j] \cup [h_j, 1]$, where $0 < g_j < h_j < 1$. Notice that $B \subset N(1, S)$.

Let p be an endpoint of the graph S (p is not an endpoint of G). Then p is an extreme of some I_i or p is an extreme of some J_j, so that $p \in B$. If p_1, p_2 are endpoints of S, then there exists an arc α in B joining p_1 and p_2. Since $\alpha \subset B \subset S \cup (\bigcup\{[0, f_i]: 1 \leqslant i \leqslant r\}) \cup (\bigcup\{[0, g_j] \cup [h_j, 1]: 1 \leqslant j \leqslant s\})$, it follows that $\alpha \subset S$. Then the unique arc in S joining p_1 and p_2 is contained in B. Since S is acyclic, then S is the union of the arcs joining its endpoints. Therefore $S \subset B$. This implies that $B \in \mathrm{RI}(\mathfrak{M}_S(t))$. Hence A is an interior point of $\mathrm{RI}(\mathfrak{M}_S(t))$. This completes the proof of (c) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a) is immediate. □

Theorem 1.3. *Suppose that G has no endpoints and $(0)_J \neq (1)_J$ for each segment J in G. If S and T are nonempty proper connected subgraphs of G such that $S \neq T$ and $\mathfrak{M}_S(t) \cap \mathfrak{M}_T(t)$ is nonempty, then the dimension of $\mathfrak{M}_S(t) \cap \mathfrak{M}_T(t)$ is not greater than* $\max\{\dim \mathfrak{M}_S(t), \dim \mathfrak{M}_T(t)\} - 2$.

Proof. Let $\mathfrak{M} = \mathfrak{M}_S(t) \cap \mathfrak{M}_T(t)$ and let $m_1 = \dim \mathfrak{M}_S(t)$ and $m_2 = \dim \mathfrak{M}_T(t)$. We may assume that T is not contained in S, then $S \subset Q(1, T)$ and $T \subset Q(1, S)$. Let I_0 be a segment in T which is not a segment of S. Then we may assume that $0 = (0)_{I_0} \in S$. If $(1)_{I_0} \in S$ or there exists another segment I_1 which belongs to T but not to S, then, by Theorem 1.1(b), $\dim \mathfrak{M} \leqslant m_1 - 2$. Thus, we may assume that I_0 is the unique segment in T which does not belong to S and $I_0 \cap S = \{0\}$. This implies that all the vertices of T except $(1)_{I_0}$ belong to S.

From the hypothesis on G, there exist two different segments I_1, I_2 in G such that $(0)_{I_1} = (1)_{I_0} = (0)_{I_2}$. Since $(1)_{I_0} \notin S$, then I_1 and I_2 do not belong to T. Thus I_1, I_2 belong to the subgraph $Q(1, T)$ but not to T. We consider three cases:

(a) The two extreme points $(1)_{I_1}$, $(1)_{I_2}$ are not in S. Then $(1)_{I_1}$, $(1)_{I_2} \notin T$ and the interior of $I_1 \cup I_2$ does not intersect $Q(1, S)$. Thus, for each $A \in \mathfrak{M}$, $A \cap I_i = (0)_{I_i}$, $i = 1, 2$. Hence, Theorem 1.1(b) implies that $\dim \mathfrak{M} \leqslant m_2 - 2$.

(b) $(1)_{I_1} \in S$ and $(1)_{I_2} \notin S$. Then $(1)_{I_2} \notin T$. Let $A \in \mathfrak{M}$, as in case (a), $A \cap I_2 = (0)_{I_2}$. Since $A \cap N(1, S)$ is connected and $(0)_{I_1} \notin N(1, S)$, then $A \cap I_1$ cannot be of the form $[(0)_{I_1}, a]$ with $a < 1$ and it cannot be of the form $[(0)_{I_1}, a] \cup [b, (1)_{I_1}]$ with $0 < a < b \leqslant 1$. Thus, either $(1)_{I_1} \in T$ or $(1)_{I_1} \notin T$, Theorem 1.1(b) implies that $\dim \mathfrak{M} \leqslant m_2 - 2$.

(c) The proof for the case $(1)_{I_1}$, $(1)_{I_2} \in S$ is similar to (b).

This completes the proof. □

Theorem 1.4. *Suppose that G has no endpoints and $(0)_J \neq (1)_J$ for each segment J in G. Let T be a maximal fine subgraph of G and let S be a proper acyclic connected subgraph of G such that $0 \leqslant \dim(\mathfrak{M}_S(t) \cap \mathfrak{M}_T(t)) = \dim \mathfrak{M}_S(t) - 1$. Then there exists a segment J in T such that $T = S \cup J$ and J does not belong to S.*

Proof. Notice that $S \neq T$. Since T is a maximal fine subgraph, every connected subgraph of G containing T properly must contain cycles, so that T is not contained in S. Let J be a segment of T which does not belong to S. Since $\mathfrak{M}_S(t) \cap \mathfrak{M}_T(t)$ is nonempty, $T \subset Q(1, S)$. Hence every segment of T intersects S. We may assume that $(0)_J \in S$. Since otherwise $\dim(\mathfrak{M}_S(t) \cap \mathfrak{M}_S(t)) \leqslant \dim \mathfrak{M}_S(t) - 2$, $(1)_J \notin S$ and J is the unique segment in T which does not belong to S. Then $T \subset S \cup J$ and $(1)_J$ is an endpoint of T. So $T - (J - \{(0)_J\})$ is a connected subgraph of G.

Suppose that there exists a segment L in S which does not belong to T. Then $(0)_L$, $(1)_L \subset T - (J - \{(0)_J\})$. Thus $(T - (J - \{(0)_J\})) \cup L$ has cycles and it is contained in S. This contradiction proves that $T = S \cup J$. □

The following lemma is easy to prove.

Lemma 1.5. *Let S and T be connected subgraphs of G such that each one of them contains all the vertices of G. Then:*

(a) $G = N(1, S) = N(1, T)$.

(b) $\mathfrak{M}_S = \{A \in C(G)\colon S \subset A\}$, $\mathfrak{M}_S(t) = \{A \in C(G)\colon S \subset A$ *and* $\mu(A) = t\}$.

(c) $\mathfrak{M}_S \cap \mathfrak{M}_T = \mathfrak{M}_{S \cup T}$ *and* $\mathfrak{M}_S(t) \cap \mathfrak{M}_T(t) = \mathfrak{M}_{S \cup T}(t)$.

Lemma 1.6. *Let $\mathfrak{A} = \mu^{-1}(t)$. Then $\mathfrak{A} = \bigcup\{\mathfrak{M}_S \cap \mathfrak{A}\colon S$ is an acyclic connected subgraph of G for which $\mu(S) < t < \mu(Q(1, S))\}$, and for each $A \in \mathfrak{A}$, $\dim_{\mathfrak{A}}(A)$ (the dimension of A in the space $\mathfrak{A}$) is equal to $\max\{\dim \mathfrak{M}_S - 1\colon S$ is an acyclic connected subgraph of G, $A \in \mathfrak{M}_S$ and $\mu(S) < t < \mu(Q(1, S))\}$.*

Proof. In [1], Duda proved that $C(X) = \bigcup\{\mathfrak{M}_S\colon S$ is an acyclic connected subgraph of $G\}$. If $\mu(S) = t$ or $\mu(Q(1, S)) = t$, then $\mathfrak{M}_S \cap \mathfrak{A}$ is a one-point set. Since $\mathfrak{A}$ has no isolated points, we conclude that $\mathfrak{A} = \bigcup\{\mathfrak{M}_S \cap \mathfrak{A}\colon S$ is an acyclic connected subgraph of G and $\mu(S) < t < \mu(Q(1, S))\}$. Then the lemma follows from Theorem 1.1(a). □

2. Large Whitney levels

Throughout this section, we will assume that G and H have no cut points and they are not simple closed curves. We also assume that $1 > t > \max \mu(\mathrm{SG}(G))$. Let $\mathfrak{A} = \mu^{-1}(t)$. Define

$v(G)$ = number of vertices of G,

$s(G)$ = number of segments of G,

$$m(G)=2(s(G)-v(G))+1,$$

$$\mathscr{S}(G)=\{S\colon S \text{ is a (not necessarily connected) subgraph of } G \text{ and } S \text{ has more than one point}\}.$$

$$\mathscr{A}(G)=\{A\in\mathfrak{A}\colon \dim_{\mathfrak{A}}(A)=m(G)\},$$

and

$$\mathscr{MF}(G)=\{S\subset G\colon S \text{ is a maximal fine subgraph of } G\}.$$

Notice that, since all the large Whitney levels are homeormophic and $m(G)$ depends only on G, $\mathscr{A}(G)$ does not depend on t.

Theorem 2.1. (a) *For each* $S\in\mathscr{MF}(G)$, $\mathfrak{M}_S(t)$ *is homeomorphic to* $I^{m(G)}$.

(b) $\dim\mathscr{A}(G)=m(G)=\dim\mathfrak{A}$.

(c) $\mathscr{A}(G)=\bigcup\{\mathfrak{M}_S(t)\colon S\in\mathscr{MF}(G)\}=\{A\in\mathfrak{A}\colon$ *there exists* $S\in\mathscr{MF}(G)$ *such that* $S\subset A\}$.

(d) *If* $K\subset\mathscr{A}(G)$ *is homeomorphic to* $I^{m(G)}$, *then there exists a unique* $S\in\mathscr{MF}(G)$ *such that* $K\subset\mathfrak{M}_S(t)$.

Proof. The proof of (a), (b) and (c) can be made with arguments similar to those in Section 7 of Duda's paper [1]. In order to prove (d), let $S_1,\ldots,S_n\in\mathscr{MF}(G)$ be such that K is contained in $\mathfrak{M}_{S_1}(t)\cup\cdots\cup\mathfrak{M}_{S_n}(t)$ and n is minimal. If $n>1$, then the set $Z=K\cap(\mathfrak{M}_{S_1}(t)\cap(\mathfrak{M}_{S_2}(t)\cup\cdots\cup\mathfrak{M}_{S_n}(t)))$ separates K. But, by Theorem 1.3, $\dim Z\leqslant m(G)-2$. This is a contradiction to the Corollary to Theorem IV.4 in [5]. Hence $n=1$ and $K\subset\mathfrak{M}_{S_1}(t)$. The uniqueness of S_1 is a consequence of Theorem 1.3. □

The structure of $\mathscr{A}(G)$ reflects important aspects of the graph G as we will see in the following result.

Theorem 2.2. $\mathscr{A}(G)$ *is homeomorphic to* $\mathscr{A}(H)$ *if and only if there exists a bijection* $\phi\colon\mathscr{S}(G)\to\mathscr{S}(H)$ *such that*:
(a) $\phi(J)$ *is a segment for each segment* J *in* G,
(b) $\phi(S)\in\mathscr{MF}(H)$ *if and only if* $S\in\mathscr{MF}(G)$, *and*
(c) *for each* $S\in\mathscr{S}(G)$, $\phi(S)=\bigcup\{\phi(J)\colon J$ *is a segment in* $S\}$.

Proof. To prove sufficiency, let $\nu\colon C(H)\to I$ be a Whitney map and let $1>t^*>\max\nu(\mathrm{SG}(H))$. Let $\mathfrak{B}=\nu^{-1}(t^*)$. So, we consider $\mathscr{A}(H)$ contained in $\mathfrak{B}$. If T is a nonempty proper connected subgraph of H, define $\mathfrak{N}_T=\{B\in C(H)\colon T\subset B\subset Q(1,T)$ and $B\cap N(1,T)$ is connected$\}$, and if $\nu(T)<t^*<\nu(Q(1,T))$, define $\mathfrak{N}_T(t^*)=\mathfrak{N}_T\cap\mathfrak{B}$.

Let $f\colon\mathscr{A}(G)\to\mathscr{A}(H)$ be a homeomorphism and let $S\in\mathscr{MF}(G)$. Theorem 2.1(b) implies $m(G)=m(H)$. From Theorem 2.1(a) and (d), there exists a unique $T\in\mathscr{MF}(H)$ such that $f(\mathfrak{M}_S(t))$ is contained in $\mathfrak{N}_T(t^*)$. Applying again Theorem 2.1, $f^{-1}(\mathfrak{N}_T(t^*))$ is contained in $\mathfrak{M}_S(t)$. Thus $f(\mathfrak{M}_S(t))=\mathfrak{N}_T(t^*)$. Define $\phi(S)=T$.

Now, we will define ϕ for a proper connected subgraph S of G which contains all the vertices of G. By definition, $\mu(S) < t$, notice that $S = S_1 \cup \cdots \cup S_m$ for some $S_1, \ldots, S_m \in \mathcal{MF}(G)$. Then

$$\begin{aligned} f(\mathfrak{M}_S(t)) &= f(\mathfrak{M}_{S_1 \cup \cdots \cup S_m}(t)) = f(\mathfrak{M}_{S_1}(t) \cap \cdots \cap \mathfrak{M}_{S_m}(t)) \\ &= f(\mathfrak{M}_{S_1}(t)) \cap \cdots \cap f(\mathfrak{M}_{S_m}(t)) = \mathfrak{N}_{\phi(S_1)}(t^*) \cup \cdots \cup \mathfrak{N}_{\phi(S_m)}(t^*) \\ &= \mathfrak{N}_{\phi(S_1) \cup \cdots \cup \phi(S_m)}(t^*). \end{aligned}$$

So, define $\phi(S) = \phi(S_1) \cup \cdots \cup \phi(S_m)$. If $T_1 \neq T_2$ are proper connected subgraphs of H which contain all the vertices of H, then $\mathfrak{N}_{T_1}(t^*) \neq \mathfrak{N}_{T_2}(t^*)$, so that ϕ is well defined.

Notice that $f(\mathfrak{M}_G(t)) = \emptyset$ and H is the unique subgraph of H which contains all the vertices of H and $\mathfrak{N}_H(t^*) = \emptyset$. Then we can define $\phi(G) = H$.

Take two proper connected subgraphs S_1, S_2 of G such that each one of them contains all the vertices of G. Let $\phi(S_1) = T_1$ and $\phi(S_2) = T_2$. Then $f(\mathfrak{M}_{S_1 \cup S_2}(t)) = f(\mathfrak{M}_{S_1}(t) \cap \mathfrak{M}_{S_2}(t)) = f(\mathfrak{M}_{S_1}(t)) \cap f(\mathfrak{M}_{S_2}(t)) = \mathfrak{N}_{T_1}(t^*) \cap \mathfrak{N}_{T_2}(t^*) = \mathfrak{N}_{T_1 \cup T_2}(t^*)$. Thus $\phi(S_1 \cup S_2) = \phi(S_1) \cup \phi(S_2)$.

Now, suppose that $S_1 \subset S_2$, then $T_1 \subset T_2$. Let m (respectively m^*) be the number of segments which belongs to S_2 but not to S_1 (respectively to T_2 but not to T_1). From Theorem 1.1(a), for $i = 1, 2$, $2(s(G) - s(S_i)) - 1 = \dim \mathfrak{M}_{S_i}(t) = \dim f(\mathfrak{M}_{S_i}(t)) = \dim(\mathfrak{N}_{T_i}(t^*)) = 2(s(H) - s(T_i)) - 1$. It follows that $m = m^*$. Notice that this equality also holds in the case that $S_i = G$ for some $i = 1, 2$.

Now we are ready to define ϕ on a segment J of G. Since G has no cut points, there exists a connected subgraph S in G which contains all the vertices of G and such that J does not belong to S (in fact, S can be chosen in $\mathcal{MF}(G)$). Since $S_1 = S \cup J$ is a connected subgraph of G which contains all the vertices of G, from the above paragraph, there exists exactly one segment L in $\phi(S_1)$ which does not belong to $\phi(S)$. Then define $\phi(J) = L$.

To prove that ϕ is well defined on J, let S_0 be another connected subgraph of G which contains all the vertices of G such that J does not belong to S_0 and let $S_2 = S_0 \cup J$. First, we will analize the case $S \subset S_0$. Then $\phi(S_2) = \phi(S_0 \cup J) = \phi(S_0 \cup S \cup J) = \phi(S_0) \cup \phi(S \cup J) = \phi(S_0) \cup \phi(S) \cup L = \phi(S_0 \cup S) \cup L = \phi(S_0) \cup L$. Thus L is the unique segment in $\phi(S_2) - \phi(S_0)$.

Suppose now that S is not contained in S_0, from the preceding paragraph, the definition of $\phi(J)$ is the same when we take S and $S \cup S_0$ and it is also the same when we take S_0 and $S \cup S_0$. Hence $\phi(J)$ is well defined.

Using the inverse map $f^{-1} : \mathscr{A}(H) \to \mathscr{A}(G)$, it is possible to define the corresponding map $\psi : \{L\colon L \text{ is a segment in } H\} \cup \{T\colon T \text{ is a subgraph of } H \text{ such that all the vertices of } H \text{ are in } T\} \to \{J\colon J \text{ is a segment in } G\} \cup \{S\colon S \text{ is a subgraph of } G \text{ such that all the vertices of } G \text{ are in } S\}$. Clearly, ψ is the inverse of ϕ.

This implies that, when J is a segment in G and S is a connected subgraph in G which contains all the vertices of G, then $J \notin S$ if and only if $\phi(J) \notin \phi(S)$.

Finally, define for $S \in \mathscr{S}(G)$, $\phi(S) = \bigcup\{\phi(J)\colon J \text{ is a segment of } S\}$. Then ϕ is the desired bijection.

Necessity: Property (c) implies that $\phi(S_1 \cup S_2) = \phi(S_1) \cup \phi(S_2)$ for every S_1, $S_2 \in \mathscr{S}(G)$. Assume that the segments in H (as those of G) are identified with the interval I. Let $m = s(G) = s(H)$. Let $J_1, \ldots, J_m$ be the segments in G. For each i, let $L_i = \phi(J_i)$. Then $L_1, \ldots, L_m$ are the segments of H. If $S \in \mathscr{M}\mathscr{F}(G)$, let $r = s(G) - s(S)$. Then $r = s(H) - s(\phi(S))$ and r does not depend on S.

Let $A \in \mathscr{A}(G)$, then A is of the form $A = \bigcup\{[(0)_{J_i}, a_i] \cup [b_i, (1)_{J_i}]\colon 1 \leqslant i \leqslant m\}$, where $0 \leqslant a_i \leqslant b_i \leqslant 1$. Define $f(A) = \bigcup\{[(0)_{L_i}, a_i] \cup [b_i, (1)_{L_i}]\colon 1 \leqslant i \leqslant m\}$. To justify that A is of this form, take an $S \in \mathscr{M}\mathscr{F}(G)$ such that $A \in \mathfrak{M}_S(t)$. Let $J_{i_1}, \ldots, J_{i_r}$ be the segments in $G - S$. Then A is of the form $A = S \cup (\bigcup\{[(0)_{J_{i_k}}, a_{i_k}] \cup [b_{i_k}, (1)_{J_{i_k}}]\colon 1 \leqslant i \leqslant r\})$. Notice that $f(A) = \phi(S) \cup (\bigcup\{[(0)_{L_{i_k}}, a_{i_k}] \cup [b_{i_k}, (1)_{L_{i_k}}]\colon 1 \leqslant i \leqslant r\})$. Since $\phi(S)$ is connected and contains all the vertices of H, then $f(A)$ is connected. We have proved that $f(A) \in C(H)$.

Clearly, f is continuous and $\mathscr{A}(G)$ is compact. Then $f(\mathscr{A}(G))$ is a compact subset of $C(H)$ which does not contain H. If $A, B \in \mathscr{A}(G)$ and $A \neq B$, since $\mu(A) = \mu(B)$, then A is not contained in B and B is not contained in A. Then there exists i such that $A \cap J_i$ is not contained in B. Thus $f(A) \cap L_i$ is not contained in $f(B)$. Hence $f(A)$ is not contained in $f(B)$. Similarly, $f(B)$ is not contained in $f(A)$. In particular, f is injective.

For each vertex p in H, consider the subgraph H_p of H which consists of all the segments in G not containing p. Since p is not a cut point of H, H_p is connected. Let $I_1, \ldots, I_M$ be the segments in H which contain p. Assume that $p = (0)_{I_i}$ for each i. For each $u \in I$, define $A_p(u) = H_p \cup (\bigcup\{[u, (1)_{I_i}]\colon 1 \leqslant i \leqslant M\})$. Then $A_p(u) \in C(H)$ and $A_p(u) \to H$ as $u \to 0$. Thus there exists $u_p > 0$ such that $u_p < 1$ and $A_p(u_p)$ is not contained in any element of $f(\mathscr{A}(G))$. Notice that each element D of the family $f(\mathscr{A}(G))$ contains p, so D is not contained in $A_p(u_p)$.

Hence, the set $\mathfrak{D} = \{A_p(u_p)\colon p \text{ is a vertex of } H\} \cup f(\mathscr{A}(G))$ is a compact subset of $C(H)$ such that no element in $\mathfrak{D}$ contains another element in $\mathfrak{D}$.

Define $\omega' : \mathfrak{D} \to I$ by $\omega'(D) = 3/4$ for each $D \in \mathfrak{D}$. From [14, Theorem 3.1], there exists a Whitney map $\omega : C(H) \to I$ which extends ω'. In order to prove that $\omega^{-1}(3/4)$ is a large Whitney level for $C(H)$, let R be a proper subgraph of H. If there exists a vertex p in H such that $p \notin R$, then R is properly contained in $A_p(u_p)$. Thus $\omega(R) < 3/4$. If all the vertices of H are contained in R, then R contains an element $T \in \mathscr{M}\mathscr{F}(H)$, then $\phi^{-1}(T) \in \mathscr{M}\mathscr{F}(G)$ and $\phi^{-1}(T) \subset \phi^{-1}(R)$. Therefore $\phi^{-1}(R)$ is a proper connected subgraph of G and since $\mu^{-1}(t)$ is a large Whitney level for $C(G)$, $\mu(\phi^{-1}(R)) < t$. Therefore, there exists $A \in \mu^{-1}(t)$ such that $\phi^{-1}(R)$ is properly contained in A. Since $\phi^{-1}(T) \subset A$, $A \in \mathscr{A}(G)$. Thus R is properly contained in $f(A)$, then $\omega(R) < 3/4$. We have proved that $\omega^{-1}(3/4)$ is a large Whitney level for $C(H)$.

It is easy to prove that $f(\mathscr{A}(G)) = \{D \in \omega^{-1}(3/4)\colon \text{there exists } T \in \mathscr{M}\mathscr{F}(G) \text{ such that } \phi(T) \subset D\} = \mathscr{A}(H)$. Hence $\mathscr{A}(G)$ is homeomorphic to $\mathscr{A}(H)$. □

Corollary 2.3. *If $\mathscr{A}(G)$ is homeomorphic to $\mathscr{A}(H)$, then H and G have the same number of segments and the same number of vertices.*

Proof. Clearly, $s(G)=s(H)$. Fix $S \in \mathscr{M}\mathscr{F}(G)$, let $K_1, \ldots, K_n$ be the segments of S, then $\phi(K_1), \ldots, \phi(K_n)$ are the segments of $\phi(S)$ and $\phi(S) \in \mathscr{M}\mathscr{F}(H)$. So $n = v(G) - 1$ and $n = v(H) - 1$. Thus $v(G) = v(H)$. □

For the necessary definitions in the following corollaries, see [15].

Corollary 2.4. *If $\mathscr{A}(G)$ is homeomorphic to $\mathscr{A}(H)$, then the cycle matroid of G is isomorphic to the cycle matroid of H.*

Proof. Let $\phi : \mathscr{S}(G) \to \mathscr{S}(H)$ is defined as in Theorem 2.2. Let $\mathscr{C}$ be a cycle of G. In order to prove the corollary, it is enough to prove that $\phi(\mathscr{C})$ is a cycle of H, see the paragraph below Theorem 1 in [15, Ch. 6].

Fix a segment J in $\mathscr{C}$. Let $S \in \mathscr{M}\mathscr{F}(G)$ be such that the graph $\mathscr{C} - J$ is contained in S. By Theorem 2.2(b), $\phi(S) \in \mathscr{M}\mathscr{F}(H)$, so $\phi(S) \cup \phi(J)$ contains a unique cycle $\mathscr{D}$ of H. Let $L \neq J$ be a segment of $\mathscr{C}$. Then the graph $(S-L) \cup J$ is connected, contains all the vertices of G and has as many segments as S. Then $(S-L) \cup J \in \mathscr{M}\mathscr{F}(G)$, so $\phi((S-L) \cup J) \in \mathscr{M}\mathscr{F}(H)$. In particular, $\phi((S-L) \cup J)$ is acyclic, and $\phi((S-L) \cup J) \cup \phi(L) = \phi(S \cup J)$ contains $\mathscr{D}$, then $\phi(L) \subset \mathscr{D}$. We have proved that $\phi(\mathscr{C}) \subset \mathscr{D}$.

Similarly, using now ϕ^{-1}, we have that $\phi^{-1}(\mathscr{D})$ is contained in a cycle of the graph $S \cup J$. Thus $\phi^{-1}(\mathscr{D}) \subset \mathscr{C}$.

Therefore, $\phi(\mathscr{C}) = \mathscr{D}$. □

Corollary 2.5. *If G and H are 3-connected and $\mathscr{A}(G)$ and $\mathscr{A}(H)$ are homeomorphic, then G and H are homeomorphic.*

Proof. It is a consequence of Corollary 2.4 and [15, Theorem 1, Ch. 6]. □

Example 2.6. It is not enough to assume that $\mathscr{A}(G)$ is homeomorphic to $\mathscr{A}(H)$ to conclude that G and H are isomorphic. Consider the graphs shown in Fig. 1. Define $\phi : \mathscr{S}(G) \to \mathscr{S}(H)$ by $\phi(S) = \bigcup\{L_i \colon J_i \text{ is a segment in } S\}$. By Theorem 2.2, $\mathscr{A}(G)$ is homeomorphic to $\mathscr{A}(H)$. However G and H are not isomorphic.

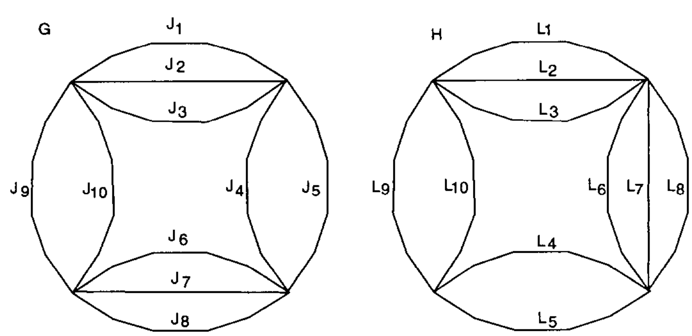

Fig. 1.

Theorem 2.7. *G and H are isomorphic graphs if and only if large Whitney levels for C(G) are homeomorphic to large Whitney levels for C(H).*

Proof. The sufficiency is immediate. For the proof of necessity, let $\mathfrak{A}$ and $\mathfrak{B}$ be respective large Whitney levels for $C(G)$ and $C(H)$. Let $f:\mathfrak{A}\to\mathfrak{B}$ be a homeomorphism. By Theorem 2.1(b), $m(G)=m(H)$. Since $\mathscr{A}(G)=\{A\in\mathfrak{A}\colon \dim_{\mathfrak{A}}(A)=m(G)\}$ and $\mathscr{A}(H)=\{B\in\mathfrak{B}\colon \dim_{\mathfrak{B}}(B)=m(H)\}$, then $f(\mathscr{A}(G))=\mathscr{A}(H)$, so that $\mathscr{A}(G)$ is homeomorphic to $\mathscr{A}(H)$. Let $\phi:\mathscr{S}(G)\to\mathscr{S}(H)$ be the bijection defined in Theorem 2.2. From [4, Theorem 8.3], in order to prove that G and H are isomorphic it is enough to show that if J and K are adjacent segments in G (where $J\neq K$), then $\phi(J)$ and $\phi(K)$ are adjacent segments in H.

Let p be a common vertex for J and K. Since p is not a cut point of G, there exists an acyclic graph S of G such that S contains all the vertices of G except p. Then $S\cup J$ and $S\cup K$ belong to $\mathscr{MF}(G)$. Thus $\phi(S\cup J)$ and $\phi(S\cup K)$ belong to $\mathscr{MF}(H)$. From Theorems 1.2 and 1.1(a), $\mathrm{RI}(\mathfrak{M}_S(t))$ is an open subset of $\mathfrak{A}$ which is homeomorphic to an open cube $(0,1)^m$ for some $m\geqslant 1$.

Notice that $\mathfrak{B}$ is of the form $\mathfrak{B}=\nu^{-1}(t^*)=\bigcup\{\mathfrak{N}_T(t^*)\colon T$ is a connected acyclic subgraph of $H\}$. Let $T_1,\dots,T_r$ be the connected acyclic subgraphs of H such that $\mathfrak{N}_{T_i}(t^*)\cap f(\mathrm{RI}(\mathfrak{M}_S(t)))\neq\emptyset$ and $\dim\,\mathfrak{N}_{T_i}(t^*)=m$. Since $f(\mathrm{RI}(\mathfrak{M}_S(t)))$ is open, it follows that it is contained in $\mathfrak{N}_{T_1}(t^*)\cup\cdots\cup\mathfrak{N}_{T_r}(t^*)$. From Theorem 1.3, $\dim(\mathfrak{N}_{T_i}(t^*)\cap\mathfrak{N}_{T_j}(t^*))\leqslant m-2$. Reasoning as in Theorem 2.1(d), $r=1$. Let $T=T_1$, then $f(\mathrm{RI}(\mathfrak{M}_S(t)))\subset\mathfrak{N}_T(t^*)$, so $f(\mathfrak{M}_S(t))\subset\mathfrak{N}_T(t^*)$. Analogously, $f^{-1}(\mathfrak{N}_T(t^*))\subset\mathfrak{M}_S(t)$. Therefore, $f(\mathfrak{M}_S(t))=\mathfrak{N}_T(t^*)$.

It is easy to prove that $\dim(\mathfrak{M}_S(t)\cap\mathfrak{M}_{S\cup J}(t))=m-1$, then $\dim(f(\mathfrak{M}_S(t))\cap f(\mathfrak{M}_{S\cup J}(t)))=m-1$. Thus $\dim(\mathfrak{N}_T(t^*)\cap\mathfrak{N}_{\phi(S\cup J)}(t^*))=m-1$. Theorem 1.4 implies that there exists a unique segment L in $\phi(S\cup J)$ such that $\phi(S\cup J)=T\cup L$ and L does not belong to T. Thus T is acyclic and L intersects T in exactly one extreme of L. This implies that T contains all the vertices of H except one extreme q. Therefore $q\in L$ and $T\subset\phi(S\cup J)$.

Similarly, $T\subset\phi(S\cup K)$. Since J does not belong to $S\cup K$, then $\phi(J)$ does not belong to $\phi(S\cup K)$. In particular, $\phi(J)$ does not belong to T. Hence $\phi(J)$ belongs to the graph $\phi(S\cup J)-T$. Thus $\phi(J)=L$. Therefore $q\in\phi(J)$. Similarly, $q\in\phi(K)$. This completes the proof of the theorem. □

3. Other properties determined by large Whitney levels

Definition 3.1. G is said to be a *fruit tree* if the following condition holds:

If γ is a simple closed curve in G, then there exists a segment J in G such that $\gamma=J$ and $(0)_J=(1)_J$.

Theorem 3.2. *G is a fruit tree if and only if large Whitney levels for C(G) are homeomorphic to cubes.*

Proof. Notice that we may assume that G is different from one interval and one simple closed curve.

To prove sufficiency, let $P = \{p \in G$: p is a vertex of G and is not an endpoint of $G\}$. Then P is a nonempty subset of G and there exists an acyclic connected proper subgraph S of G such that $P \subset S$ and S does not contain endpoints of G. Since G is a fruit tree, each element in P is a cut point of G. Let $\mathfrak{A} = \mu^{-1}(t)$ be a large Whitney level for $C(G)$. Then, for each $A \in \mathfrak{A}$, $P \subset A$. Applying again that G is a fruit tree, it follows that $S \subset A$. This implies that $\mathfrak{A} = \mathfrak{M}_S(t)$. Therefore, $\mathfrak{A}$ is homeomorphic to a cube.

Necessity: Let $\mathfrak{A} = \mu^{-1}(t)$ be a large Whitney level for $C(G)$. Define $V' = \{p$: p is a vertex of G and p is not an endpoint of $G\}$, $\mathscr{MF}'(G) = \{S$: S is an acyclic subgraph of G such that the vertices of S are precisely the elements in $V'\}$. Notice that $\mathscr{MF}'(G)$ is nonempty, and for each $S \in \mathscr{MF}'(G)$, $Q(1, S) = G$ and $\mathfrak{M}_S(t)$ is nonempty. If $S \in \mathscr{MF}'(G)$, the number of segments in S is equal to the number of points in V' minus one; the number of segments in G which meet S in exactly one point is equal to the number of endpoints of G and the number of segments which meet S in their two extremes is equal to $s(G)$ − (the number of endpoints of G) − (the number of segments in S). Therefore, the dimension of $\mathfrak{M}_S(t)$ does not depend on S. Reasoning as in Section 7 in [1], $\mathscr{A}'(G) = \{A \in \mathfrak{A}$: $\dim_{\mathfrak{A}} A = \dim \mathfrak{A}\} = \bigcup\{\mathfrak{M}_S(t)$: $S \in \mathscr{MF}'(G)\}$.

Let S_1, $S_2 \in \mathscr{MF}'(G)$ such that $S_1 \neq S_2$. Then there exists a segment J in S_1 such that J is not a segment of S_2. Then the extremes of J are points in S_1. Each element in $\mathfrak{M}_{S_1}(t) \cap \mathfrak{M}_{S_2}(t)$ contains J. Theorem 1.1(b) implies that $\dim(\mathfrak{M}_{S_1}(t) \cap \mathfrak{M}_{S_2}(t)) \leqslant \dim \mathfrak{M}_{S_1}(t) - 2$.

Since we are assuming that $\mathfrak{A}$ is homeomorphic to a cube, then $\mathscr{A}'(G) = \mathfrak{A}$. Reasoning as in Theorem 2.1(d), there exists a unique element S in $\mathscr{MF}'(G)$ and $\mathfrak{A} = \mathfrak{M}_S(t)$.

Suppose that G is not a fruit tree, then there exists a simple closed curve γ in G such that γ is the union of more that one segment of G. Notice that all the vertices in the graph γ belong to S. Since S is acyclic, there exists a segment J in γ such that J is not a segment of S. Let α be the unique arc joining $(0)_J$ and $(1)_J$ in S and let L be any segment in α. Then $S_1 = J \cup (S - (L - \{(0)_L, (1)_L\}))$ is connected, it contains all the elements of V' and it has as many segments as S. This implies that S_1 is a tree. Hence $S_1 \in \mathscr{MF}'(G) - \{S\}$. This contradiction proves that G is a fruit tree and completes the proof of the theorem. □

The following theorem is a consequence of Theorems A and D in [7].

Theorem 3.3. *G is a simple closed curve if and only if large Whitney levels for G are not unicoherent.*

Remark 3.4. Suppose that G has a cut point p. Let $\mathfrak{A}$ be a large Whitney level for $C(G)$, then p belongs to each element in $\mathfrak{A}$. Thus, Lynch's theorem in [11] implies that $\mathfrak{A}$ is an AR. Therefore $\mathfrak{A}$ is contractible.

Question 3.5. Is it true that if large Whitney levels for $C(G)$ are contractible, then G has cut points?

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